ON THE CONSEQUENCES OF DRUCKER'S POSTULATE FOR PLASTIC ANISOTROPIC MEDIA

(O SLEDSTVIIAKH POSTULATA DRUKKERA DLIA PLASTICHESKIKH ANIZOTROPNYKH SRED)

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The paper investigates the limits imposed by Drucker's postulate [1 and 2] on possible yield limits in tension or compression for anisotropic plastic media.

Let us consider an anisotropic elastic-plastic body; whether the anisotropy exists from the beginning or whether it is a result of certain deformation process, will be of no importance. We shall assume that the mean pressure does not affect the plastic properties of the material.

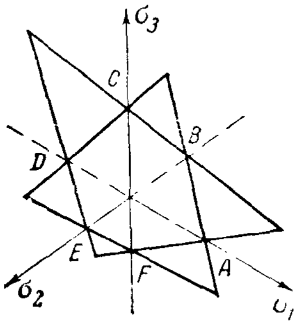

Fig. 1

Suppose that the body is under the action of a certain system of body forces and surface loads which give rise to a state of stress σ_{ij}^* within the body. Suppose also that a further loading from some external action is applied to the body and then removed. Drucker's postulate [1 and 2] requires that the work done by the external forces during loading is positive and that the work done by the external forces over the whole cycle of loading and unloading is not negative. Let σ_{ij} represent the state of stress in the body after the application of the external forces and ε_{ij} the rate of plastic deformations. It follows from Drucker's postulate that

$$(\sigma_{ij} - \sigma_{ij}^*)\varepsilon_{ij} \geqslant 0 \qquad (1)$$

If the yield surface is given in the form

$$\varphi(\sigma_{ij}) = 1 \qquad (2)$$

then it follows from (1) that in the six-dimensional space the surface (2) will not be concave and the rate of plastic deformations will be given by the associated flow law

$$\varepsilon_{ij} = \frac{\partial\varphi}{\partial\sigma_{ij}} \qquad (3)$$

The yield surface of any plastically anisotropic body which is insensitive to hydrostatic pressure can be written in the following form for certain fixed point in time

$$f(\sigma_1 - \sigma_2, \sigma_1 - \sigma_3, \sigma_2 - \sigma_3, l_i, m_i, n_i) = 1 \qquad (4)$$

where l_i, m_i, n_i are the direction cosines of the principal directions as shown in the Table.

	1	2	3
x	l_1	m_1	n_1
y	l_2	m_2	n_2
z	l_3	m_3	n_3

If two of the principal stresses are equal, then in the plane in which these principal stresses lie all directions will be principal directions, and therefore the plasticity condition need not depend on the direction cosines of these directions, i.e. the following equalities must hold:

$$\partial f / \partial n_i = \partial f / \partial m_i = 0 \quad \text{for } \sigma_2 = \sigma_3$$

$$\partial f / \partial l_i = \partial f / \partial m_i = 0 \quad \text{for } \sigma_1 = \sigma_2$$

$$\partial f / \partial l_i = \partial f / \partial n_i = 0 \quad \text{for } \sigma_1 = \sigma_3$$

If the yield limits in tension and compression are known in three mutually orthogonal directions, then in the deviation plane of Fig.1 six points are known on the yield curve. It follows from the condition of non-concavity of the yield surface that all the possible plasticity conditions lie between the hexagon $ABCDEF$ and the outer six-cornered star obtained by forming the sides of the hexagon, and that the plasticity condition in the form of the hexagon $ABCDEF$ proposed in [3] defines the lower bound for the load combination at which the body passes into a plastic state.

In order to determine the range of variation of possible plasticity conditions we must determine the restrictions imposed by Drucker's postulate on the yield limits in tension and compression in every possible direction, i.e. the restrictions imposed on the functions $k(\alpha_i)$ and $s(\alpha_i)$, where α_i are the cosines of the angles between the direction of the tension (or compression) and the coordinate axes x, y and z.

In the case of pure tension in the direction (l_1, l_2, l_3) the following state of stress is set up in the body:

$$\sigma_x = k(l_i)\, l_1^2, \qquad \sigma_y = k(l_i)\, l_2^2, \qquad \sigma_z = k(l_i)\, l_3^2$$
$$\tau_{xy} = k(l_i)\, l_1 l_2, \qquad \tau_{xz} = k(l_i)\, l_1 l_3, \qquad \tau_{yz} = k(l_i)\, l_2 l_3 \tag{5}$$

Suppose that as a result of some external forces the state of stress $\sigma_x = k(1, 0, 0)$, $\sigma_y = \sigma_z = \tau_{xy} = \tau_{yz} = \tau_{xz} = 0$, is produced in the body and that before the application of the external forces the body was elastic and in a state of stress sufficiently close to that defined by (5) for certain given values of l_i. Suppose also the the external forces are shch that they cause the body to pass from the state close to (5) to the state $\sigma_x = k(1, 0, 0)$, $\sigma_y = \sigma_z = \tau_{xy} = \tau_{yz} = \tau_{xz} = 0$ only in an elastic manner. In this case the inequality (1) may be written in the form

$$[k(1, 0, 0) - k(l_i)\, l_1^2]\, \varepsilon_x - k(l_i)\, l_2^2\, \varepsilon_y - k(l_i)\, l_3^2\, \varepsilon_z -$$
$$- k(l_i)\, l_1 l_2\, \varepsilon_{xy} - k(l_i)\, l_1 l_3\, \varepsilon_{xz} - k(l_i)\, l_2 l_3\, \varepsilon_{yz} \geqslant 0 \tag{6}$$

where ε_x, ε_y, ε_z, ε_{xy}, ε_{xz} and ε_{yz} are the rates of plastic deformations for the state of stress

$$\sigma_x = k(1, 0, 0), \quad \sigma_y = \sigma_z = \tau_{xz} = \tau_{yz} = \tau_{xy} = 0$$

In order to determine deformation rate at this point we apply the associated flow law.

Since $\sigma_1 - \sigma_3 = (\sigma_1 - \sigma_2) + (\sigma_2 - \sigma_3)$, the plasticity condition (4) can always be reduced to the form

$$f(\sigma_1 - \sigma_2, \sigma_2 - \sigma_3, l_i, m_i, n_i) = 1$$

Having solved the plasticity condition for $\sigma_1 - \sigma_2$ we ezpand this solution in a series in powers of $(\sigma_1 - \sigma_3)$

$$\sigma_1 - \sigma_2 = k(l_i) - A(\sigma_2 - \sigma_3) + \ldots \qquad \left(A = \frac{\partial f}{\partial(\sigma_2 - \sigma_3)} \Big/ \frac{\partial f}{\partial(\sigma_1 - \sigma_2)}\right) \tag{7}$$

Here we set the expression $\sigma_2 - \sigma_3 = 0$, $\sigma_1 - \sigma_2 = k(l_i)$ for A and in this way A will be a function only of the direction cosines l_1, m_1, n_1.

Condition (7) may be written in the form

$$\frac{\sigma_1 - \sigma_2}{k(l_i)} + \frac{A}{k(l_i)}(\sigma_2 - \sigma_3) + \ldots = 1 \tag{8}$$

Applying the associated yield law (8), we obtain

$$\varepsilon_{ij} = \lambda\left[k^{-1}(l_\alpha)\left(\frac{\partial\sigma_1}{\partial\sigma_{ij}} - \frac{\partial\sigma_2}{\partial\sigma_{ij}}\right) + (\sigma_1 - \sigma_2)\frac{\partial k^{-1}}{\partial l_\alpha}\frac{\partial l_\alpha}{\partial\sigma_{ij}} + Ak^{-1}(l_\alpha)\left(\frac{\partial\sigma_2}{\partial\sigma_{ij}} - \frac{\partial\sigma_3}{\partial\sigma_{ij}}\right) + \right.$$
$$\left. + (\sigma_2 - \sigma_3)\left(\frac{\partial Ak^{-1}}{\partial l_\alpha}\frac{\partial l_\alpha}{\partial\sigma_{ij}} + k^{-1}(l_\alpha)\frac{\partial A}{\partial m_\alpha}\frac{\partial m_\alpha}{\partial\sigma_{ij}} + k^{-1}(l_\alpha)\frac{\partial A}{\partial n_\alpha}\frac{\partial n_\alpha}{\partial\sigma_{ij}}\right)\right] + \ldots \tag{9}$$

The stress components are related to the principal stresses and the direction cosines by Formulas

$$\sigma_{ij} = \sigma_1 l_i l_j + \sigma_2 m_i m_j + \sigma_3 n_i n_j \tag{10}$$

Only three of the direction cosines are independent, since they are related as follows:

$$l_i l_j + m_i m_j + n_i n_j = \delta_{ij} \tag{11}$$

Differentiating Equations (10) and (11) with respect to σ_{ij} and substituting the values of the direction cosines

$$l_1 = m_2 = n_3 = 1, \quad l_2 = l_3 = m_1 = m_3 = n_1 = n_2 = 0,$$

we obtain, after simple rearrangement,

$$\frac{\partial\sigma_1}{\partial\sigma_x} = \frac{\partial\sigma_2}{\partial\sigma_y} = \frac{\partial\sigma_3}{\partial\sigma_z} = 1, \qquad \frac{\partial\sigma_\alpha}{\partial\sigma_{ij}} = 0$$
$$\frac{\partial l_2}{\partial\tau_{xy}} = -\frac{\partial m_1}{\partial\tau_{xy}} = \frac{1}{\sigma_1 - \sigma_3}, \qquad \frac{\partial l_3}{\partial\tau_{xz}} = -\frac{\partial n_1}{\partial\tau_{xz}} = \frac{1}{\sigma_1 - \sigma_2} \tag{12}$$
$$\frac{\partial m_3}{\partial\tau_{yz}} = -\frac{\partial n_2}{\partial\tau_{yz}} = \frac{1}{\sigma_2 - \sigma_3}, \qquad \frac{\partial l_\alpha}{\partial\sigma_{ij}} = \frac{\partial m_\alpha}{\partial\sigma_{ij}} = \frac{\partial n_\alpha}{\partial\sigma_{ij}} = 0$$

Substituting the values given by (12) into equalities (9), we find that

$$\varepsilon_x = \lambda + \ldots, \qquad \varepsilon_y = \lambda(A - 1) + \ldots, \qquad \varepsilon_z = -\lambda A + \ldots$$
$$\varepsilon_{xy} = \lambda\left[-\frac{1}{k}\frac{\partial k}{\partial l_2} + \frac{\sigma_2 - \sigma_3}{\sigma_1 - \sigma_3}\frac{\partial}{\partial l_2}\frac{A}{k} + \ldots\right]$$
$$\varepsilon_{xz} = \lambda\left[-\frac{1}{k}\frac{\partial k}{\partial l_3}\frac{\sigma_1 - \sigma_2}{\sigma_1 - \sigma_3} + \frac{\sigma_2 - \sigma_3}{\sigma_1 - \sigma_2}\frac{\partial}{\partial l_3}\frac{A}{k} + \ldots\right] \tag{13}$$
$$\varepsilon_{yz} = \lambda\left[\frac{\partial A}{\partial m_3} - \frac{\partial A}{\partial n_2} + \ldots\right]$$

Setting $\sigma_2 = \sigma_3$ in equalities (13) we obtain

$$\varepsilon_x = \lambda, \qquad \varepsilon_y = \lambda(A - 1), \qquad \varepsilon_z = -\lambda A$$
$$\varepsilon_{xy} = -\frac{\lambda}{k}\frac{\partial k}{\partial l_2}, \qquad \varepsilon_{xz} = \frac{\lambda}{k}\frac{\partial k}{\partial l_3}, \qquad \varepsilon_{yz} = \lambda\left(\frac{\partial A}{\partial m_2} - \frac{\partial A}{\partial n_2}\right) \tag{14}$$

If the point $\sigma_2 = \sigma_3$ is a point of intersection of regular surfaces, then by representing all these surfaces in the form (8) and making use of the concept of a generalized plastic potential [4], we obtain equality (14) in the form

$$\varepsilon_x = \lambda_1 + \ldots + \lambda_n, \qquad \varepsilon_y = \lambda_k (A_k - 1), \qquad \varepsilon_z = \lambda_k A_k$$

$$\varepsilon_{xy} = -\frac{\lambda_1 + \ldots + \lambda_n}{k} \frac{\partial k}{\partial l_2}, \qquad \varepsilon_{xz} = -\frac{\lambda_1 + \ldots + \lambda_n}{k} \frac{\partial k}{\partial l_3}$$

$$\varepsilon_{yz} = \lambda_k \left(\frac{\partial A_k}{\partial m_3} - \frac{\partial A_k}{\partial n_2} \right)$$

Also $\lambda_1, \ldots, \lambda_n \geqslant 0$. Substituting the values obtained into inequality (6), we find that

$$(k_0 - kl_1^2)(\lambda_1 + \ldots + \lambda_n) + kl_2^2 \lambda_k (1 - A_k) + kl_3^2 \lambda_k A_k +$$
$$+ \frac{k}{k_0} \left(\frac{\partial k_0}{\partial l_2} l_1 l_2 + \frac{\partial k_0}{\partial l_3} l_1 l_3 \right)(\lambda_1 + \ldots + \lambda_n) - kl_2 l_3 \lambda_k \left(\frac{\partial A_k}{\partial m_2} - \frac{\partial A_k}{\partial n_2} \right) \geqslant 0 \qquad (15)$$

Here the quantities with the index 0 are evaluated at the point

$$l_1 = m_2 = n_3 = 1.$$

Setting all λ_k with the exception of λ_1 equal to zero, we obtain

$$k_0 - kl_1^2 - (A_i - 1) kl_2^2 + A_i k l_3 + l_1 l_3 \frac{k}{k_0} \frac{\partial k_0}{\partial l_2} +$$
$$+ l_1 l_2 \frac{k}{k_0} \frac{\partial k_0}{\partial l_3} + \left[\frac{\partial A_i}{\partial n_2} - \frac{\partial A_i}{\partial m_3} \right] kl_2 l_3 \geqslant 0 \qquad (16)$$

We will show that the quantity in square brackets in Equation (16) can be set equal to zero by the choice of n_2 and m_3.

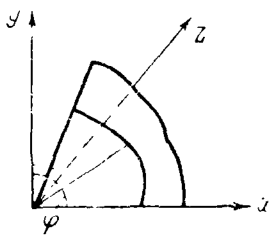

Fig. 2

The quantity A_1 is the coefficient of $\sigma_2 - \sigma_3$ in the plasticity condition (7). If we reverse the directions of axes 2 and 3, condition (7) must not be altered, and consequently

$$A_i (m_3, n_2) = A_i (-m_3, -n_2)$$

Instead of m_3 and n_2 we introduce new variables $\xi = m_3 - n_2$ and $\eta = m_3 + n_2$. Then

$$\frac{\partial A_i}{\partial n_2} - \frac{\partial A_i}{\partial m_3} = -2 \frac{\partial A_i}{\partial \xi}$$

and $\partial A_1 / \partial \xi$ vanishes for certain value of ξ, since $A(\xi) = A_1(-\xi)$. By a choice of m and n, condition (16) can now be reduced to the form

$$k_0 - kl_1^2 + (1 - A_i) kl_2^2 + A_i kl_3^2 +$$
$$+ \frac{k}{k_0} \frac{\partial k_0}{\partial l_2} l_1 l_2 + l_1 l_2 \frac{k}{k_0} \frac{\partial k_0}{\partial l_3} \geqslant 0 \qquad (17)$$

Supposing that until the application of the additional loading $l_2 = l_3$ or $l_2 = -l_3$, we establish that $k(l_1)$ must be such that

$$k_0 - kl_1^2 + kl^2 + kl_1 l \frac{1}{k_0} \frac{\partial k_0}{\partial l_2} + kl_1 l \frac{1}{k_0} \frac{\partial k_0}{\partial l_3} \geqslant 0 \qquad \left(l = \frac{l_2 + l_3}{2} \right)$$
$$k_0 - kl_1^2 + kl_*^2 + kl_1 l_* \frac{1}{k_0} \frac{\partial k_0}{\partial l_2} - kl_1 l_* \frac{\partial k_0}{\partial l_3} \geqslant 0 \qquad \left(l_* = \frac{l_2 - l_3}{2} \right) \qquad (18)$$

There are no terms in the inequalities (18) which are related to the form of the plasticity condition, and these inequalities determine the limitations on the results of possible experimental values of the yield limits in all possible directions for materials obeying Drucker's postulate.

If we take into account that

$$\frac{\partial k_0}{\partial l_2}+\frac{\partial k_0}{\partial l_3}=\frac{\partial k_0}{\partial l}, \qquad \frac{\partial k_0}{\partial l_2}-\frac{\partial k_0}{\partial l_3}=\frac{\partial k_0}{\partial l_*}$$

then the inequalities (18) assume the form

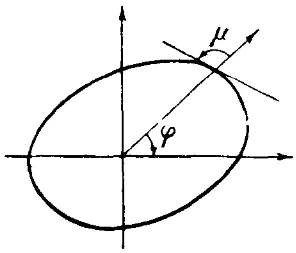

Fig. 3

$$k_0-k+3kl^2+k\sqrt{1-2l^2}\,\frac{l}{k_0}\frac{\partial k_0}{\partial l}\geqslant 0$$

$$k_0-k+3kl_*^2+k\sqrt{1-2l_*^2}\,\frac{l_*}{k_0}\frac{\partial k_0}{\partial l}\geqslant 0 \qquad (19)$$

The two inequalities (19) are equivalent and their slight difference in form is due solely to the choice of the initial system of coordinates; in future we shall use only one of them.

In order to formulate conditions (19) locally, we assume that the external force system is sufficiently small and that we can take it as close to zero as we like. Expanding the left-hand side of (19) in a series in l, letting l tend to zero and dividing both sides of the inequality by l^2, we obtain

$$\frac{d^2k}{dl^2}+6k+\frac{1}{k}\left(\frac{\partial k}{\partial l}\right)^2\geqslant 0 \qquad (20)$$

We introduce the angle φ in the plane $l_2 = l_3$, as shown in Fig.2. Then

$$l={}^1/_2\sin\varphi$$

Inequality (20) now assumes the form

$$\frac{d^2k}{d\varphi^2}+3k+\frac{1}{k}\left(\frac{dk}{d\varphi}\right)^2\geqslant 0 \qquad (21)$$

If we make an appropriate choice of coordinate system x, y, z, it is not difficult to see that the inequality (21) holds in any plane, and since on rotation through any angle the form of inequality (21) remains unaltered, it follows that it holds not only for $\varphi = 0$ but for any angle φ.

Let us consider some plane in the body and select a direction in this plane as the axis of a system of polar coordinates ρ, φ. For each value of the angle φ we measure ρ as the yield limit in the direction defined by this angle. We thus obtain the curve shown in Fig.3, the equation for which is

$$\rho = k(\varphi) \qquad (22)$$

Inequality (21) requires that

$$\sin^3\mu + 3\sin\mu \geqslant \varkappa \qquad (23)$$

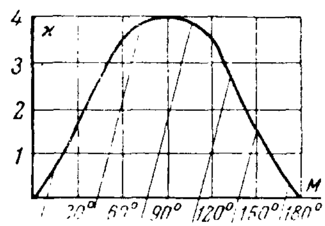

Fig. 4

where μ is the angle between the direction of the tangent to the curve and the radius vector (Fig.3) and $\varkappa = k(\varphi)/R$ is the nondimensional curvature of the curve. In Fig.4 the field of variation of the parameters $\varkappa$ and μ is shown hatched.

Thus Drucker's postulate imposes restrictions on the curves (22), which are convex relative to the origin of coordinates, but on segments where these curves are concave, inequality (23) is satisfied for any $\mu > 0$.

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